over the surface above the source will yield sufficiently precise information. The source temperature in this case is ≈10% lower than for the single-layer Si structure.

The foregoing results confirm the legitimacy of simplified calculations neglecting the influence of the layers and treating the heat-conduction problem in the crystal of a semiconductor IC as in a homogeneous Si domain. Experiments on the source temperature from the surface of an IC having an $Si - SiO_2 - Al$ structure yield excessively low results.

Analogous calculations of the temperatures on the faces of the structure $Si - SiO_2 - Al$ for $Bi = 0.75 \cdot 10^{-3}$ show that the external heat-transfer rate has virtually no effect on the relief of the temperature field.

NOTATION

$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, Laplace operator; $\theta_r(x, y, z) = T_r(x, y, z) - T_{me}$; $T_r(x, y, z)$, temperature in the r-th layer; T_{me}, temperature of medium; λ_i^0, δ_i, thermal conductivity and thickness of i-th layer; $\psi = P/\lambda_0^0 V$; P, power of local source; $V = 2l_1 \times 2l_2 \times h$; e(x), unit Heaviside function; α, heat-transfer coefficient; ε, η, center coordinates of source; k number of layers covering the source.

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SOLUTION OF THE UNSTEADY HEAT-CONDUCTION EQUATION IN AN INHOMOGENEOUS MEDIUM

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The solution of an unsteady two-dimensional heat-conduction problem in an inhomogeneous medium is investigated by using differential operators.

If there are no heat sources or sinks within a body, the unsteady two-dimensional heat-conduction problem is described by the equation

$$c\gamma \frac{\partial T}{\partial t} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{\partial \lambda}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial \lambda}{\partial y}\frac{\partial T}{\partial y}, \quad (1)$$

where the thermal conductivity $\lambda = \lambda(x, y)$, the density $\gamma = \gamma(x, y)$, and the specific heat $c = c(x, y)$ are given functions of the coordinates x and y.

We seek the solution of Eq. (1) which satisfies appropriate boundary conditions [1] and has the form

$$T = \tau(t)\,\Psi(x, y). \quad (2)$$

Substituting (2) into (1) and introducing the separation of variables parameter $-\nu^2$, we obtain the two equations

$$\frac{d\tau}{dt} = -\nu^2\tau; \quad (3)$$

$$\Delta\Psi + \frac{1}{\lambda}\,\mathrm{grad}\,\Psi\,\mathrm{grad}\,\lambda + \frac{c\gamma\nu^2}{\lambda}\,\Psi = 0, \quad (4)$$

where Δ is the two-dimensional Laplacian.

Hence it follows that the solution of Eq. (1) can be written in the form

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$$T(x, y, t) = \sum_{\nu=1}^{\infty} A_\nu e^{-\nu^2 t} \Psi_\nu(x, y). \tag{5}$$

We solve Eq. (4) by Bergman's method of linear operators in differential form [2, 3].

Assuming that the parameters λ, c, and γ are functions only of the variable x, we construct the solution of (4) in the form

$$\Psi(x, y) = \sum_{n=0}^{\infty} d_n(x) \Phi^{(n)}(z). \tag{6}$$

Here Φ(z) is an arbitrary function of the complex variable $z = x + iy$. The real and imaginary parts of (6) are solutions, and so is a linear combination of them, since Eq. (4) is linear.

We determine the real coefficients $d_n(x)$ from the condition that (6) satisfies Eq. (4). Substituting (6) into (4) we obtain

$$\sum_{n=0}^{\infty} \left[\left(d_n'' + \frac{\lambda'}{\lambda} d_n' + \frac{\nu^2 c\gamma}{\lambda} d_n \right) \Phi^{(n)} + \left(2d_n' + \frac{\lambda'}{\lambda} d_n \right) \Phi^{(n+1)} \right] = 0. \tag{7}$$

The arbitrary function Φ(z) will convert Eq. (7) to an identity if we require the coefficients $d_n(x)$ to satisfy the conditions

$$d_0'' + \frac{\lambda'}{\lambda} d_0' + \frac{\nu^2 c\gamma}{\lambda} d_0 = 0; \tag{8}$$

$$d_n'' + \frac{\lambda'}{\lambda} d_n' + \frac{\nu^2 c\gamma}{\lambda} d_n = -\left(2d_{n-1}' + \frac{\lambda'}{\lambda} d_{n-1} \right) \quad (n = 1, 2, \ldots). \tag{9}$$

If, e.g., we specify or approximate the coefficients λ, c, and γ by power functions

$$\lambda = ax^p; \quad c\gamma = bx^q \quad (a > 0, b > 0), \tag{10}$$

Eq. (8) reduces to Bessel's equation

$$x^2 d_0'' + pxd_0' + \delta^2 \nu^2 x^{p-q+2} d_0 = 0 \quad (\delta^2 = b/a). \tag{11}$$

Its solution is [4]

$$d_0 = x^{(1-p)/2} Y_s \left(\frac{2\delta\nu}{q-p+2} x^{(q-p+2)/2} \right) \quad (s = (1-p)/(q-p+2)). \tag{12}$$

Here Y_s is a linear combination of Bessel functions of the first and second kind.

Using the properties of Bessel functions for $p = q = 2$, it is not difficult to obtain the following expressions for the first two coefficients d_0 and d_1:

$$a_0 = \frac{B_0}{x} \sin \delta\nu (x + b_0);$$

$$a_1 = \frac{B_1}{x} \sin \delta\nu (x + b_1) - B_0 \sin \delta\nu (x + b_0). \tag{13}$$

If Φ(z) in (6) is given in the form of a power function

$$\Phi(z) = z^k, \tag{14}$$

where k is a positive integer, series (6) will contain a finite number of terms, and there is no question about its convergence. For example, let $k = 1$. Then Eq. (6) takes the form

$$\Psi(x, y) = a_0(x) z + a_1(x). \tag{15}$$

Forming the sum of the real and imaginary parts of function (15), taking account of (13) for $b_0 = b_1 = 0$, and using (5), we obtain

$$T(x, y, t) = \frac{B_1 + B_0 y}{x} \sum_{\nu=1}^{\infty} A_\nu e^{-\nu^2 t} \sin \delta\nu x. \tag{16}$$

If $B_0 = 1$, $B_1 = -h$, $\delta = 2\pi/l$, solution (16) will satisfy the special boundary conditions

$$T(x, y, \infty) = T(x, h, t) = T(l, y, t) = 0;$$

$$T(x, y, 0) = \varphi(x, y) = \frac{y-h}{x} f(x). \tag{17}$$

Here h and l are parameters defining the dimensions of the body and f(x) is a given function which can be expanded in the interval $(0, l)$ in a Fourier sine series with the argument $\delta\nu x$.

In general, it is convenient to choose the function $\Phi(z)$ in (6) in the form of a complex Fourier series

$$\Phi(z) = \sum_{n=1}^{\infty} (C_n e^{-n\delta z} + D_n e^{n\delta z}),$$

where the coefficients C_n and D_n are determined so that solution (6) satisfies the boundary conditions.

Approximate solutions of Eq. (4), and consequently also of (1), can be treated when λ, c, and γ depend on the two coordinates x and y.

Setting $\nu = 1$ in (4), we seek the solution of this equation in the form

$$\Psi(x, y) = a(x, y)\Phi(z). \tag{18}$$

Substituting (18) into (4), we obtain the expression

$$\left(\Delta a + \frac{1}{\lambda} \operatorname{grad}\lambda \operatorname{grad} a + \frac{c\gamma}{\lambda} a\right)\Phi + \left[2\left(\frac{\partial a}{\partial x} + i\frac{\partial a}{\partial y}\right) + \frac{a}{\lambda}\left(\frac{\partial \lambda}{\partial x} + i\frac{\partial \lambda}{\partial y}\right)\right]\Phi' = 0. \tag{19}$$

The arbitrary function $\Phi(z)$ satisfies this equation only when the conditions

$$\Delta a + \frac{1}{\lambda} \operatorname{grad}\lambda \operatorname{grad} a + \frac{c\gamma}{\lambda} a = 0, \tag{20}$$

$$2\frac{\partial a}{\partial x} + \frac{a}{\lambda}\frac{\partial \lambda}{\partial x} = 0; \quad 2\frac{\partial a}{\partial y} + \frac{a}{\lambda}\frac{\partial \lambda}{\partial y} = 0 \tag{21}$$

are satisfied.

Substituting into (20) $a = D/\sqrt{\lambda}$ found from (21), we obtain

$$\Delta\lambda - \frac{1}{2\lambda}[\operatorname{grad}\lambda]^2 + 2c\gamma = 0$$

or

$$\lambda^{1/2}\Delta\lambda^{1/2} + c\gamma = 0. \tag{22}$$

If, e.g., we set

$$c\gamma = b(x^2 + y^2)^p \lambda^{1/2}, \tag{23}$$

the solution of (22) will have the form

$$\lambda^{1/2} = F(z) - \frac{b(x^2 + y^2)^{p+1}}{4(p+1)^2}, \tag{24}$$

where F(z) is an arbitrary analytic function of the complex argument $z = x + iy$. The arbitrariness in the function (23) and (24) can be used to approximate the given or experimentally determined spatial dependences of λ, c, and γ.

NOTATION

T, temperature; t, time; x, y, linear coordinates; λ, thermal conductivity; γ, density; c, specific heat; $T(x, y, t)$, $\tau(t)$, $\Psi(x, y)$, $d_n(x)$, and $a(x, y)$, unknown functions; $\varphi(x)$, f(x), given functions; $\Phi(z)$, arbitrary function; A_ν, B_0, b, b_0, b_1, C_n, D_n, and B_1, undetermined constants.

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